Gravitational Wave Radiation by Binary Black Holes

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1 INTRODUCTION

For over 100 years, Albert Einstein's general theory of relativity has remained the prevailing description of gravitation. It is one of the most successful theories in physics and has made remarkable predictions that have been observed and confirmed with incredible accuracy. One such prediction, made by Einstein himself shortly after publishing his theory of relativity, is the existence of gravitational waves - although he never thought that they could be detected. In an interesting series of events, Einstein actually came to doubt his prediction and attempted to publish a paper with Nathan Rosen arguing that gravitational waves could not exist. His paper was rejected in the peer-review process on the basis of an error in the calculation. Einstein was furious that his paper was even shown to reviewers before being published, never-mind the "erroneous" claims that he had made an error, and vowed to never submit another paper to Physical Review. Einstein later confirmed the error and resubmitted his paper (to a different publisher), this time providing an argument supporting the existence of gravitational waves [1]. In 2005 it was revealed in a search through the records at the Physical Review that the until then anonymous referee who rejected Einstein's paper was one Howard P. Robertson, the same Robertson of which the famous metric in cosmology shares part of its name.

The properties of gravitational waves, should they even describe a physical process rather than be merely a mathematical construct (as Rosen thought), has been a difficult and hotly debated topic permeating the decades after their postulation by Einstein. Richard Feynman was the first to claim that gravitational waves could transport energy, using his famous "sticky bead argument" [1]. If the waves could carry energy, then in theory one could devise an experiment that could measure them. Since then many attempts at measuring gravitational waves have been made, and have largely been unsuccessful (not unsurprisingly, as their effects are exceptionally small) until quite recently.

In this paper we will derive the formulae describing gravitational waves from the Einstein field equations, as Einstein himself had done, as well as derive several important features of gravitational radiation. Namely, we shall discover the quadrupole formula describing the energy contained within the gravitational waves and apply that formula to a system of two black holes in orbit about one another. Such a black hole binary is the precise configuration that produced the gravitational waves detected by the twin LIGO observatories in the United States just this last year: the first direct detection of gravitational waves.

2 The Linearized Theory of Gravitational Waves

We first consider the case of gravitational waves propagating through empty spacetime. To capture the effects of these waves on the surrounding spacetime, we write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric describing our background spacetime, and $h_{\mu\nu}$ is the perturbation to the metric induced by the gravitational waves. A powerful and important assumption to make is that the perturbation is very small, i.e. $\left|h_{\mu\nu}\right| \ll 1$. To first order we may raise and lower indices using the background metric, so that we may define the inverse metric $g^{\mu\nu}$ to be

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{2.2}$$
$$g_{\mu\nu}g^{\mu\sigma} = \left(\eta_{\mu\nu} + h_{\mu\nu}\right)\left(\eta^{\mu\sigma} - h^{\mu\sigma}\right) = \delta^\sigma_\nu - h^\sigma_\nu + h^\sigma_\nu - h_{\mu\nu}h^{\mu\sigma} \approx \delta^\sigma_\nu.$$

With this assumption in mind, we may immediately begin calculating the Christoffel symbols

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right), \qquad \partial_\rho g_{\mu\nu} \equiv \frac{\partial g_{\mu\nu}}{\partial x^\rho}. \tag{2.3}$$

Using the metric (2.1) and its inverse (2.2), we have

$$\begin{aligned}\Gamma^\sigma_{\mu\nu} &= \frac{1}{2}\left(\eta^{\sigma\rho} - h^{\sigma\rho}\right)\left[\partial_\mu(\eta_{\rho\nu} + h_{\rho\nu}) + \partial_\nu(\eta_{\rho\mu} + h_{\rho\mu}) - \partial_\rho(\eta_{\mu\nu} + h_{\mu\nu})\right] \\ &= \frac{1}{2}\left(\partial_\mu h^\sigma_\nu + \partial_\nu h^\sigma_\mu - \eta^{\sigma\rho}\partial_\rho h_{\mu\nu}\right) - \frac{1}{2}h^{\sigma\rho}\left(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}\right).\end{aligned} \tag{2.4}$$

The Riemann curvature tensor is defined by

$$R^\sigma_{\rho\mu\nu} = \partial_\mu\Gamma^\sigma_{\rho\nu} - \partial_\nu\Gamma^\sigma_{\rho\mu} + \Gamma^\sigma_{\alpha\mu}\Gamma^\alpha_{\rho\nu} - \Gamma^\sigma_{\alpha\nu}\Gamma^\alpha_{\rho\mu}, \tag{2.5}$$

and the Ricci curvature tensor by

$$R_{\rho\nu} = R^\sigma_{\rho\sigma\nu}. \tag{2.6}$$

Putting the Christoffel symbols (2.4) into (2.6), we have to first order[1]

$$R_{\rho\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\rho h^\sigma_\nu + \partial_\nu\partial_\sigma h^\sigma_\rho - \partial_\sigma\partial^\sigma h_{\rho\nu} - \partial_\nu\partial_\rho h\right) \tag{2.7}$$

where we have written

$$h = h^\sigma_\sigma. \tag{2.8}$$

Before we write out the full Einstein equations, there are a number of simplifications we can make. First, define the trace-reversed perturbation

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{2.9}$$

We notice that under a change of coordinates given by

$$x'^\mu = x^\mu + \xi^\mu, \tag{2.10}$$

the trace-reversed perturbation transforms as [2]

$$\tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\sigma\xi^\sigma. \tag{2.11}$$

If we require that ξ^μ satisfies

$$\partial_\sigma\partial^\sigma\xi^\mu = -\partial_\sigma\tilde{h}^{\sigma\mu}, \tag{2.12}$$

[1] See Appendix A for the derivation, including terms up to second order.

then we find

$$\partial_\nu \tilde{h}'^{\mu\nu} = 0. \tag{2.13}$$

This condition (2.13) is called the Lorenz gauge[2] [2], which can be equivalently written

$$\partial_\nu h^{\mu\nu} = \frac{1}{2}\partial^\mu h. \tag{2.14}$$

This allows us to greatly simplify the expression (2.7) for the Ricci tensor. We find

$$\begin{aligned} R_{\rho\nu} &= \frac{1}{2}\left(\partial_\sigma\partial_\rho h^\sigma_\nu + \partial_\nu\partial_\sigma h^\sigma_\rho - \partial_\sigma\partial^\sigma h_{\rho\nu} - \partial_\nu\partial_\rho h\right) \\ &= \frac{1}{2}\left[\cancel{\partial_\rho\left(\frac{1}{2}\partial_\nu h\right)} + \cancel{\partial_\nu\left(\frac{1}{2}\partial_\rho h\right)} - \partial_\sigma\partial^\sigma h_{\rho\nu} - \cancel{\partial_\nu\partial_\rho h}\right] \\ &= -\frac{1}{2}\partial_\sigma\partial^\sigma h_{\rho\nu}. \end{aligned} \tag{2.15}$$

The Ricci scalar is simply

$$R \equiv \eta^{\rho\nu} R_{\rho\nu} = -\frac{1}{2}\partial_\sigma\partial^\sigma h, \tag{2.16}$$

and thus the Einstein tensor is

$$G_{\rho\nu} \equiv R_{\rho\nu} - \frac{1}{2}\eta_{\rho\nu} R = -\frac{1}{2}\partial_\sigma\partial^\sigma\left(h_{\rho\nu} - \frac{1}{2}\eta_{\rho\nu} h\right) = -\frac{1}{2}\partial_\sigma\partial^\sigma \tilde{h}_{\rho\nu}. \tag{2.17}$$

Finally, the Einstein field equations read

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.18}$$

or

$$\boxed{\Box \tilde{h}_{\mu\nu} = -16\pi T_{\mu\nu},} \tag{2.19}$$

in geometrized units ($G = c = 1$) where $\Box = \partial_\sigma\partial^\sigma$ is the d'Alembert operator. In a vacuum, Eq. (2.19) is the familiar wave equation

$$\boxed{\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\tilde{h}_{\mu\nu} = 0,} \tag{2.20}$$

and Eq. (2.19) is the wave equation with a source. In other words, *the trace-reversed perturbation satisfies a wave equation*! This is the phenomena of gravitational waves: ripples in spacetime propagating through space at the speed of light. The simplest solution of (2.20) is a plane wave [3]

$$\tilde{h}_{\mu\nu} = A_{\mu\nu} e^{i k_\alpha x^\alpha}, \quad k_\alpha = (\omega, k_1, k_2, k_3) \tag{2.21}$$

with frequency ω. $A_{\mu\nu}$ is a 4x4 symmetric constant tensor giving the amplitudes of each component of the wave [4], and is also sometimes called the polarization tensor [3]. The Lorenz gauge condition on the plane wave solution gives

$$\begin{aligned} \partial_\nu \tilde{h}^{\mu\nu} = \partial_\nu\left(A^{\mu\nu} e^{i k_\alpha x^\alpha}\right) &= i k_\nu A^{\mu\nu} e^{i k_\alpha x^\alpha} = 0 \\ &\to k_\nu A^{\mu\nu} = 0. \end{aligned} \tag{2.22}$$

So, the amplitudes are orthogonal to the direction of oscillation, i.e., the waves are *transverse*. Eq. (2.20) provides a further freedom to set any four components of the $\tilde{h}_{\mu\nu}$ equal to zero [4]. By far the most common and convenient choice of these are

$$\tilde{h}^{0\mu} = 0 \qquad \text{(transverse)}$$
$$\tilde{h}^\mu_\mu = 0 \qquad \text{(traceless)}$$

[2]The majority of textbooks (with the exception of [2]) incorrectly label this as the Lorentz gauge. Beware of the confusion.

For this reason, this particular choice of coordinates is called the transverse-traceless (TT) gauge [5]. An immediate result of using the transverse-traceless gauge is that Eq. (2.9) gives

$$\tilde{h}_{\mu\nu}^{\mathrm{TT}} = h_{\mu\nu}^{\mathrm{TT}}. \tag{2.23}$$

So the trace-reversed perturbation coincides with the metric perturbation in these particular coordinates. The Lorenz and TT gauge use up all of the coordinate-freedoms, leaving the plane wave solution with only two independent components such that

$$A_{\mu\nu} = h_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + h_\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.24}$$

This reveals the two independent polarizations of gravitational waves: plus (+) and cross (×). The coefficients h_+, $h_\times$ are the amplitudes of each respective polarization and may be any function of $t - z$ for waves propagating in the z direction. The general solution is a linear combination of the two. Also of interest is the vacuum wave equation for the plane wave solution, which reads

$$\Box \tilde{h}_{\mu\nu} = k_\alpha k^\alpha \tilde{h}_{\mu\nu} = 0 \ \rightarrow \ k_\alpha k^\alpha = 0 \tag{2.25}$$

I.e., k_α is null, or $\omega^2 = k^2$; the wave has velocity equal to that of light.

3 Source Emission of Gravitational Waves

We now turn to the problem of solving Eq. (2.19), the Einstein equations for a linearized gravitational wave to first order with a source:

$$\partial_\sigma \partial^\sigma \tilde{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{3.1}$$

The approach to obtaining a solution is as follows: we will first derive an expression for $\tilde{h}_{\mu\nu}$ to first order by solving Eq. 3.1, and then use that solution in a second iteration through the Einstein equations up to second order to determine the energy contained in the waves themselves. This approach is justified by the fact that this approximation is valid to an error of the order of the amplitude of the gravitational waves, but for the concept of a gravitational wave to make any sense they must be of incredibly small scales such that for all practical purposes the errors are negligible. Later we will see that the energy contained in a gravitational wave must be thought of in an averaged, “smeared-out” sense, which will introduce an error on the order of the ratio of the reduced wavelength of the gravitational wave ($\lambda/2\pi$) to the background radius of curvature. Again, such an error is so small that for any conceivable purpose it may be neglected [5].

Focusing only on waves emitted by the source (i.e., excluding “ingoing” waves), Eq. 3.1 is solved by

$$\tilde{h}_{\mu\nu}(t, \vec{x}) = 4 \int_{\text{all space}} \frac{T_{\mu\nu}\left(t - |\vec{x} - \vec{x}'|\right)}{|\vec{x} - \vec{x}'|} d^3 x', \tag{3.2}$$

where $\vec{x}$ is the position 3-vector, $|\vec{x} - \vec{x}'|$ denotes taking the magnitude of the difference of the position vectors, $d^3 x' = dx'^1 dx'^2 dx'^3$, and the stress-energy-momentum tensor $T_{\mu\nu}$ is taken to be evaluated at the retarded time[3] $t - |\vec{x} - \vec{x}'|$. The form of such a solution is well-known from the study of wave equations with sources, as is true in the analogy with electrodynamics in the emission of electromagnetic radiation by sources [6]. In order to solve (3.2), we will make the following simplifying assumption commonly known as the “slow-motion approximation” [5]:

[3] In $c \neq 1$ units, the retarded time has the form $t - |\vec{x} - \vec{x}'|/c$.

For a source contained within a radius R_{source} with mean internal velocity v and angular velocity $\omega = v/R_{\text{source}}$, then the "characteristic reduced wavelength" $\lambda_{\text{red}} = \lambda/2\pi = 1/\omega$ of the radiation is assumed to be long compared to the source's size:

$$\lambda_{\text{red}} \gg R_{\text{source}} \quad \text{or} \quad R_{\text{source}}/\lambda_{\text{red}} \ll 1.$$

This is the same as the statement

$$v \ll 1.$$

In the case of binary black holes orbiting at ~ 0.1c, as we will later investigate, this approximation turns out to be excellent.

Defining our coordinate system to be centered on the source with the origin contained within R_{source}, then at large radial distances from the source

$$r = |\vec{x}| \gg R_{\text{source}} = \left|\vec{x}'\right|. \tag{3.3}$$

The integral (3.2) may then be expanded in powers of $\vec{x}'/r$, making use of the slow-motion approximation [5]:

$$\tilde{h}_{\mu\nu}(t,\vec{x}) = \frac{4}{r}\int T_{\mu\nu}\left(t-r,\vec{x}'\right) d^3x' + \mathcal{O}\left[\frac{x^j}{r^2\lambda_{\text{red}}}\int x'^j\, T_{\mu\nu}\left(t-r,\vec{x}'\right) d^3x'\right]. \tag{3.4}$$

Note that in the transverse-traceless (TT) gauge, only the spatial components of $\tilde{h}_{\mu\nu}$, $\tilde{h}_{ij}$, are nonzero. Recall also that the stress-energy-momentum tensor has the conservation law

$$\partial_\nu T^{\mu\nu} = 0. \tag{3.5}$$

From this conservation law we may construct several useful identities [5]:

$$\begin{aligned} &\partial_0\partial_\nu T^{\mu\nu} = 0 \\ \to\ &\partial_0\partial_0 T^{\mu 0} + \partial_0\partial_i T^{\mu i} = 0. \end{aligned} \tag{3.6}$$

In particular for $\mu = 0$:

$$\partial_0\partial_0 T^{00} = -\partial_0\partial_i T^{0i}, \tag{3.7}$$

$$= -\partial_i\partial_0 T^{i0}. \tag{3.8}$$

(3.7) and (3.8) are merely related by the symmetry of $T^{\mu\nu}$ and the fact that partial derivatives commute. Further, consider

$$\begin{aligned} &\partial_\mu\partial_\nu T^{\mu\nu} = 0 \\ \to\ &\partial_0\partial_\nu T^{0\nu} + \partial_i\partial_\nu T^{i\nu} = 0 \\ \to\ &\partial_0\partial_0 T^{00} + \partial_0\partial_i T^{0i} + \partial_i\partial_0 T^{i0} + \partial_i\partial_j T^{ij} = 0 \\ \to\ &\partial_0\partial_0 T^{00} - 2\partial_0\partial_0 T^{00} + \partial_i\partial_j T^{ij} = 0 && \text{(by (3.7) and (3.8))} \\ \to\ &\partial_0\partial_0 = \partial_i\partial_j T^{ij}. \end{aligned} \tag{3.9}$$

Now, consider the quantity

$$\begin{aligned} \partial_k\partial_l\left(T^{kl}x^ix^j\right) &= \partial_k\left(\partial_l T^{kl}x^ix^j + T^{kl}(\delta^i_l x^j + \delta^j_l x^i\right) \\ &= \partial_k\partial_l T^{kl}x^ix^j + \partial_l T^{kl}(\delta^i_k x^j + \delta^j_l x^i) + \partial_k\left(T^{ki}x^j + T^{kj}x^i\right). \end{aligned} \tag{3.10}$$

Consider for a moment the second term. By the product rule

$$\begin{aligned} &\partial_l\left[T^{kl}\left(\delta^i_k x^j + \delta^j_k x^i\right)\right] = \partial_l T^{kl}\left(\delta^i_k x^j + \delta^j_k x^i\right) + T^{kl}\left(\delta^i_k\delta^j_l + \delta^j_k\delta^i_l\right) \\ \to\ &\partial_l T^{kl}\left(\delta^i_k x^j + \delta^j_k x^i\right) = \partial_l\left(T^{il}x^j + T^{jl}x^i\right) - 2T^{ij}. \end{aligned} \tag{3.11}$$

Thus (3.10) becomes

$$\partial_k\partial_l\left(T^{kl}x^ix^j\right)=\partial_k\partial_l T^{kl}x^ix^j+2\partial_k\left(T^{ki}x^j+T^{kj}x^i\right)-2T^{ij}. \tag{3.12}$$

Finally, consider the expression

$$\begin{aligned}\partial_0\partial_0\left(T^{00}x^ix^j\right)&=\partial_0\partial_0 T^{00}x^ix^j\\ &=\partial_k\partial_l T^{kl}x^ix^j && \text{(by (3.9))}\\ &=\partial_k\partial_l\left(T^{kl}x^ix^j\right)-2\partial_k\left(T^{ki}x^j+T^{kj}x^i\right)+2T^{ij}. && \text{(by (3.12))}\end{aligned}$$

We define the second moment of the mass distribution according to [5] as

$$I^{ij}(t)\equiv\int T^{00}x^ix^j\,d^3x. \tag{3.13}$$

We may then rewrite (by (3.12)) as

$$\begin{aligned}\frac{\partial^2 I^{ij}}{\partial t^2}=\frac{d^2I^{ij}}{dt^2}&=\partial_0\partial_0\int T^{00}x^ix^j\,d^3x\\ &=\int\partial_0\partial_0\left(T^{00}x^ix^j\right)d^3x\\ &=\int\partial_k\partial_l\left(T^{kl}x^ix^j\right)d^3x-2\int\partial_k\left(T^{ki}x^j+T^{kj}x^i\right)d^3x+2\int T^{ij}d^3x.\end{aligned} \tag{3.14}$$

The first two integrals may be rewritten using Gauss's theorem as surface integrals in which we take the surface out to infinity. Assuming the source is bounded so that T^{ij} is only defined in some finite region of space near the origin, these surface integrals vanish, and we are left with

$$\int T^{ij}d^3x=\frac{1}{2}\frac{d^2I}{dt^2}. \tag{3.15}$$

Then, by substitution of (3.15) into (3.4) and dropping the higher order terms (i.e., we assume the observer is at a distance much larger than the reduced wavelength $r\gg\lambda_{\text{red}}\gg R_{\text{source}}$), we have

$$\tilde{h}_{ij}(t,\vec{x})=\frac{2}{r}\frac{d^2I_{ij}(t-r)}{dt^2}. \tag{3.16}$$

The quantities I_{ij}, however, are not able to be directly computed by an external observer [5]. However, if we define the reduced quadrupole moment in terms of the second mass moment by

$$Q_{ij}\equiv I_{ij}-\frac{1}{3}\delta_{ij}I^k_k=\int T^{00}\left(x^ix^j-\frac{1}{3}\delta_{ij}r^2\right)d^3x, \tag{3.17}$$

where $r^2=x_ix^i$, then conveniently Q_{ij} and I_{ij} are identical in the TT gauge [5]. We may then write[4]

$$\boxed{h^{\text{TT}}_{ij}(t,\vec{x})=\frac{2}{r}\ddot{Q}^{\text{TT}}_{ij}(t-r)} \tag{3.18}$$

where we recall in the TT gauge $\tilde{h}^{\text{TT}}_{ij}=h^{\text{TT}}_{ij}$, and the reduced quadrupole moment is taken to be evaluated at the retarded time $t-r$. Eq. (3.18) is the famous quadrupole formula, first derived by Einstein in 1918 [3]. Thus the gravitational waves emitted by some slow-moving source of mass-energy-density $T^{00}=\rho$ is proportional to the second time derivative of the source's quadrupole moment. The quadrupole moment may be calculated in exactly the same way as one would in Newtonian theory; as the coefficient in the quadrupole term of a multipole expansion of the Newtonian potential [2]. Of course, it makes sense that gravitational radiation must be of at least quadrupole nature or higher: gravitational monopoles cannot exist by conservation of mass, and gravitational dipoles cannot exist by conservation of momentum (i.e., there is no such thing as a negative mass). In electromagnetism, however, there may be dipole radiation because there do exist positive and negative electric charges.

[4] For even more compactness, we have introduced the dot notation for differentiation with respect to time: $\dot{x}\equiv dx/dt$.

4 The Energy Contained in Gravitational Waves

As Richard Feynman's "sticky-bead experiment" had argued, gravitational waves, like their electromagnetic analogs, may transport and deposit energy to their surroundings. This has many huge physical implications. For example, a system emitting gravitational waves like a binary star or black hole system is effectively *losing* energy via the production of gravitational radiation. First, this implies that the binary's orbital velocity must decrease, by conservation of energy, and so eventually the two bodies will merge! Second, the radiation leaving the binaries may transport that energy to Earth, where we may construct some form of detector to measure the energy in these gravitational waves and identify such an event. As gravitational waves travel unimpeded through the universe, this presents a wealth of possibilities for astronomers to utilize gravitational radiation in observing the universe beyond what is possible in the electromagnetic spectrum; for example, "seeing" beyond the cosmic microwave background radiation and into the earliest moments of the universe.

In this section we will derive an expression for the energy contained in a gravitational wave. We begin by returning to the Einstein field equations:

$$G_{\rho\nu} \equiv R_{\rho\nu} - \frac{1}{2} g_{\rho\nu} R = 8\pi T_{\rho\nu}. \tag{4.1}$$

In the previous section, we solved these equations for a linearized gravitational wave to first order to obtain an expression for the gravitational waves h_{ij}. In only considering the equations to first order, we have ignored the possibility that gravitational waves themselves are a source of curvature. To capture this, we must consider the equations to second order. Mimicking the notation used in [2], we write the Ricci tensor as a sum of a component only involving first order contributions from $h_{\mu\nu}$ and a component involving terms of second order:

$$R_{\rho\nu} = R^{(1)}_{\rho\nu} + R^{(2)}_{\rho\nu}, \tag{4.2}$$

where $R^{(1)}_{\rho\nu}$ is given by Eq. (2.7) and $R^{(2)}_{\rho\nu}$ is given by[5]

$$\begin{aligned} R^{(2)}_{\rho\nu} = \frac{1}{2}\Big(& h^{\sigma\lambda}\partial_\rho\partial_\nu h_{\sigma\lambda} + h^{\sigma\lambda}\partial_\sigma\partial_\lambda h_{\rho\nu} - h^{\sigma\lambda}\partial_\lambda\partial_\nu h_{\rho\sigma} - h^{\sigma\lambda}\partial_\lambda\partial_\rho h_{\nu\sigma} \\ & + \partial_\sigma h^{\sigma\lambda}\partial_\lambda h_{\rho\nu} + \partial^\sigma h^\lambda_\nu \partial_\sigma h_{\lambda\rho} - \partial^\sigma h^\lambda_\nu \partial_\lambda h_{\sigma\rho} - \partial_\sigma h^{\sigma\lambda}\partial_\nu h_{\rho\lambda} - \partial_\sigma h^{\sigma\lambda}\partial_\rho h_{\lambda\nu}\Big) \\ + \frac{1}{4}\Big(& \partial_\rho h_{\sigma\lambda}\partial_\nu h^{\sigma\lambda} + \partial_\lambda h \partial_\rho h^\lambda_\nu + \partial_\lambda h \partial_\nu h^\lambda_\rho - \partial_\lambda h \partial^\lambda h_{\rho\nu}\Big). \end{aligned} \tag{4.3}$$

This allows us to write Eq. (4.1) as

$$\left(R^{(1)}_{\rho\nu} + R^{(2)}_{\rho\nu}\right) - \frac{1}{2}\eta_{\rho\nu}\left(R^{(1)} + R^{(2)}\right) = 8\pi T_{\rho\nu}. \tag{4.4}$$

Moving some terms around, we rewrite Eq. (4.4) as

$$R^{(1)}_{\rho\nu} - \frac{1}{2}\eta_{\rho\nu} R^{(1)} = 8\pi\left(T_{\rho\nu} + t_{\rho\nu}\right), \tag{4.5}$$

where we have defined

$$t_{\rho\nu} \equiv -\frac{1}{8\pi}\left(R^{(2)}_{\rho\nu} - \frac{1}{2}\eta_{\rho\nu} R^{(2)}\right). \tag{4.6}$$

Eq. (4.5) is the same linearized equation we solved to obtain Eq. (3.18), except that now we have added an extra term to the right hand side that depends on the second order effects of the metric perturbation. In the way that we have written Eq. (4.5), it is as if the gravitational waves are themselves a source of curvature. The quantities $t_{\rho\nu}$ are conventionally called the energy-momentum pseudotensor of the gravitational field, and describe the effects of the gravitational waves themselves on the curvature of the spacetime surrounding them.

[5] See Appendix A for the derivation.

There are several undesirable properties of the pseudotensor; namely, as the name suggests, it is not a tensor, and it is also not invariant under gauge transformations [2]. The former is not particularly an issue, and the latter may be overcome by instead thinking of the energy-momentum pseudotensor as averaged over a spacetime volume of sides larger than the wavelength of the gravitational waves. The notation used in this averaging is universally denoted by angle brackets,

$$t_{\rho\nu} = \left\langle -\frac{1}{8\pi}\left(R^{(2)}_{\rho\nu} - \frac{1}{2}\eta_{\rho\nu}R^{(2)}\right)\right\rangle. \tag{4.7}$$

The averaging may be thought of as "smearing out" the energy-momentum of the gravitational waves over enough of a region (i.e., larger than λ, but smaller than the radius of the background curvature) that the curvature induced by the gravitational waves may be physically described in a gauge-invariant manner [2]. The smaller the wavelength of the gravitational wave is in comparison to the background curvature, the better this approximation becomes, becoming arbitrarily accurate in the limit of a very distant observer in an asymptotically flat spacetime [4]. For a source emitting gravitational radiation with a well-defined cyclic period, this amounts to simply averaging over the period of the source.

To derive an expression for $t_{\rho\nu}$, we must calculate Eq. (4.6) given Eq. (4.3), and then average according to Eq. (4.7). We find, in the TT gauge[6],

$$\boxed{t_{\rho\nu} = \frac{1}{32\pi}\left\langle \partial_\rho h^{\mathrm{TT}}_{\sigma\lambda}\partial_\nu h^{\sigma\lambda}_{\mathrm{TT}}\right\rangle.} \tag{4.8}$$

If one were to conduct the derivation without assuming a gauge, they would find

$$t_{\rho\nu} = \frac{1}{32\pi}\left\langle \partial_\rho h_{\sigma\lambda}\partial_\nu h^{\sigma\lambda} - \frac{1}{2}\partial_\rho h\partial_\nu h - \partial_\sigma h^{\sigma\lambda}\partial_\rho h_{\nu\lambda} - \partial_\sigma h^{\sigma\lambda}\partial_\nu h_{\rho\lambda}\right\rangle, \tag{4.9}$$

which in fact happens to be gauge invariant [2]. Using the conditions of the TT gauge, it can be seen that (4.9) does in fact reduce to (4.8), as the second term includes a trace and the last two divergences, and so only the first term survives.

We now have a gauge-invariant quantity that takes a relatively simple form in a particular choice (TT gauge) and describes the energy stored in the gravitational waves. We can utilize this to determine the luminosity, or energy per unit time transported in the waves. We can determine this from the energy flux t_{0r} in the radial direction, averaged over the surface of a sphere far away from and containing the source. First we find an expression for t_{0r} using Eq. (4.8) and Eq. (3.18). Noting that the quadrupole moment only depends on the time $t-r$, we can use the chain rule to write

$$\begin{aligned} t_{0r} &= \frac{1}{32\pi}\left\langle \partial_0 h^{\mathrm{TT}}_{\rho\sigma}\partial_r h^{\rho\sigma}_{TT}\right\rangle \\ &= \frac{1}{32\pi}\left\langle \left(\frac{2}{r}\dddot{Q}^{\mathrm{TT}}_{ij}\right)\left(-\frac{2}{r}\dddot{Q}^{ij}_{\mathrm{TT}} - \frac{2}{r^2}\ddot{Q}^{ij}_{\mathrm{TT}}\right)\right\rangle \\ &\approx -\frac{1}{8\pi r^2}\left\langle \dddot{Q}^{\mathrm{TT}}_{ij}\dddot{Q}^{ij}_{\mathrm{TT}}\right\rangle, \end{aligned} \tag{4.10}$$

where in the last step we have used the fact that we are considering the waves at a distance very far from the source. Unfortunately, we might not necessarily know how to explicitly express the quadrupole moment in the TT gauge. We can, however, calculate the regular reduced quadrupole moment via (3.17) quite easily. Thus we want to express (4.10) in terms of Q_{ij}, not Q^{TT}_{ij}. We may do so using the projection tensor defined in [2],

$$P_{ij} = \delta_{ij} - n_i n_j, \tag{4.11}$$

where n_i is a normal vector in the i direction and δ_{ij} is the usual Kronecker delta = 1 if $i = j$ and 0 otherwise. The transverse-traceless part of a tensor may then be expressed in terms of the projection

[6]See Appendix B for the derivation.

tensor and the original tensor X_{ij} as [2]

$$X_{ij}^{\mathrm{TT}} = \left(P_i^k P_j^l - \frac{1}{2} P_{ij} P^{kl}\right) X_{kl}, \tag{4.12}$$

which allows us to write, after a considerable amount of tedious algebra (involving 64 terms of Kronecker deltas),

$$\dddot{Q}_{ij}^{\mathrm{TT}} \dddot{Q}_{\mathrm{TT}}^{ij} = \dddot{Q}_{ij} \dddot{Q}^{ij} - 2\dddot{Q}_i^j \dddot{Q}^{ik} n_j n_k + \frac{1}{2} \dddot{Q}^{ij} \dddot{Q}^{kl} n_i n_j n_k n_l. \tag{4.13}$$

The power radiated by the gravitational waves can now be calculated from

$$\begin{aligned}
\frac{dE}{dt} &= \int_{S^2_\infty} t_{0r} r^2 d\Omega \\
&= -\frac{1}{8\pi} \int_{S^2_\infty} \left\langle \dddot{Q}_{ij}^{\mathrm{TT}} \dddot{Q}_{\mathrm{TT}}^{ij} \right\rangle d\Omega \\
&= -\frac{1}{8\pi} \int_{S^2_\infty} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} - 2\dddot{Q}_i^j \dddot{Q}^{ik} n_j n_k + \frac{1}{2} \dddot{Q}^{ij} \dddot{Q}^{kl} n_i n_j n_k n_l \right\rangle d\Omega,
\end{aligned} \tag{4.14}$$

where the integral is taken over a sphere at infinity and $d\Omega = \sin\theta d\theta d\phi$ is the infinitesimal solid angle. Since the quadrupole moments only depend on $t - r$, they are constant on the surface of the sphere and may be taken outside of the integral:

$$\frac{dE}{dt} = -\frac{1}{8\pi} \left[\left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle \int_{S^2_\infty} d\Omega - 2 \left\langle \dddot{Q}_i^j \dddot{Q}^{ik} \right\rangle \int_{S^2_\infty} n_j n_k d\Omega + \frac{1}{2} \left\langle \dddot{Q}^{ij} \dddot{Q}^{kl} \right\rangle \int_{S^2_\infty} n_i n_j n_k n_l d\Omega \right]. \tag{4.15}$$

Using the identities for averages over the surface of a sphere [2, 5, 6],

$$\begin{aligned}
\int d\Omega &= 4\pi, \\
\int n_i n_j d\Omega &= \frac{4\pi}{3} \delta_{ij}, \\
\int n_i n_j n_k n_l d\Omega &= \frac{4\pi}{15} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right),
\end{aligned} \tag{4.16}$$

we may simplify Eq. (4.15) as follows:

$$\begin{aligned}
\frac{dE}{dt} &= -\frac{1}{8\pi} \left[4\pi \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle - \frac{8\pi}{3} \left\langle \dddot{Q}_i^j \dddot{Q}^{ik} \right\rangle (\delta_{jk}) + \frac{2\pi}{15} \left\langle \dddot{Q}^{ij} \dddot{Q}^{kl} \right\rangle \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \right] \\
&= -\frac{1}{8\pi} \left[4\pi \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle - \frac{8\pi}{3} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle + \frac{2\pi}{15} \left(\left\langle \dddot{Q}^{ij} \dddot{Q}^{kl} \right\rangle \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \right) \right] \\
&= -\frac{2\pi}{8\pi} \left[\left(2 - \frac{4}{3}\right) \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle + \frac{1}{15} \left\langle \dddot{Q}\dddot{Q} + 2\dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle \right] \\
&= -\frac{1}{4} \left(2 - \frac{4}{3} + \frac{2}{15}\right) \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle \qquad \text{(since } Q_{ij} \text{ is traceless)} \\
&= -\frac{1}{5} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle.
\end{aligned} \tag{4.17}$$

Thus, the luminosity of the gravitational waves is

$$\boxed{L_{\mathrm{GW}} = -\frac{dE}{dt} = \frac{1}{5} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle.} \tag{4.18}$$

We now have an expression for the energy emitted by a slow-moving matter distribution in the form of gravitational radiation! It is proportional to the square of the third time derivative of the quadrupole moment of that matter distribution. In the next section, we will utilize this formula to determine the energy lost in a binary system of black holes and compare our theoretical prediction to the measurements made by the LIGO observatory.

5 Gravitational Radiation by Binary Black Holes

We shall follow the construction of a binary black hole system as presented in [7]. Consider a system consisting of two black holes of unequal masses M_1 and M_2 locked together in orbit about one another. Placing the origin of our coordinates at the barycenter of the two masses, we define the spacial position vectors of the two black holes as $\vec{r}_1(t)$ and $\vec{r}_2(t)$, as well as the velocities of the two bodies $\vec{v}_1$ and $\vec{v}_2$. The magnitudes of these vectors are denoted simply be the same variable without a vector arrow. Let the orbit of the black holes be in the $x-y$ plane. Since both stars must make a complete orbit in the same amount of time, which we denote as the period P, we have

$$P = \frac{v_1}{2\pi r_1} = \frac{v_2}{2\pi r_2} \Rightarrow r_1 = \frac{v_1}{v_2} r_2. \tag{5.1}$$

Balancing the centripetal force with the Newtonian gravitational force for the first black hole, which is reasonable to do in the non-relativistic limit on which the slow-motion approximation was developed, we find the relation

$$\frac{M_1 v_1^2}{r_1} = \frac{M_1 M_2}{(r_1 + r_2)} \Rightarrow v_1^2 = \frac{M_1}{(r_1 + r_2)^2} r_2. \tag{5.2}$$

and likewise for v_2. Recall that we are still working in units in which $G = c = 1$. Dividing the expressions for v_1 and v_2 and substituting (5.1), we find that $r_1 = (M_2/M_1) r_2$. Then defining the separation vector $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, which has length $r = r_1 + r_2$, we have

$$\vec{r}_1 = \frac{M_2}{M_1 + M_2}\vec{r}, \quad \vec{r}_2 = -\frac{M_1}{M_1 + M_2}\vec{r}. \tag{5.3}$$

Likewise, we can rewrite the velocities in a similar manner using (5.1):

$$\vec{v}_1 = \frac{M_2}{M_1 + M_2}\vec{v}, \quad \vec{v}_2 = -\frac{M_1}{M_1 + M_2}\vec{v}, \tag{5.4}$$

where here $\vec{v} \equiv \vec{v}_1 - \vec{v}_2$ is the relative velocity vector. A schematic relating these basic properties of the orbital configuration is given in Figure 5.1. A convenient quantity to define is the symmetric mass ratio[7], which is given by

$$\eta \equiv \frac{M_1 M_2}{(M_1 + M_2)^2}. \tag{5.5}$$

[7] Do not confuse this η with the contraction of the Minkowski metric with itself, $\eta = \eta^{\mu\nu}\eta_{\mu\nu}$, which equals four.

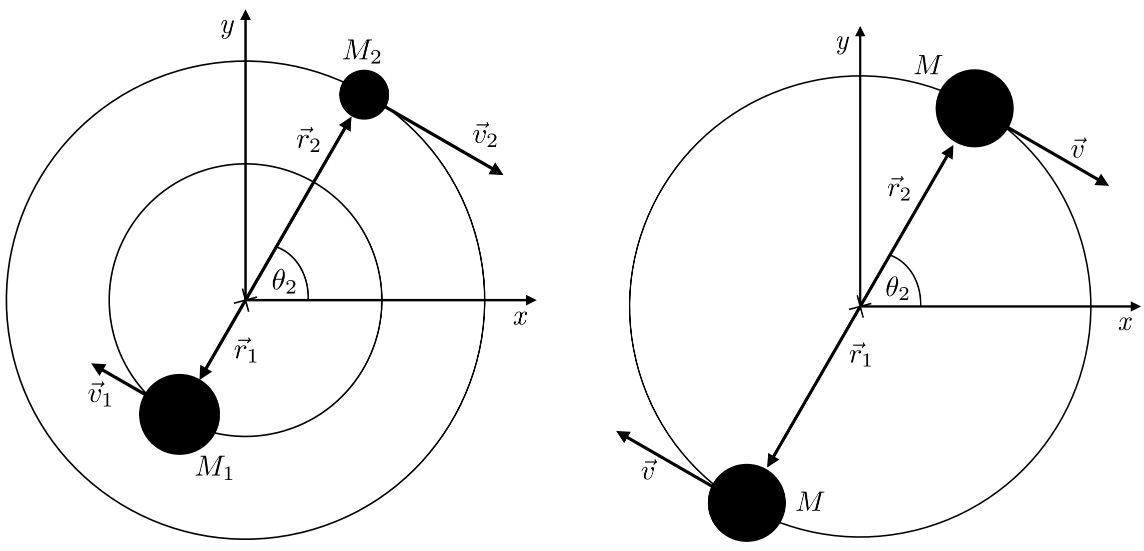

Figure 5.1: Schematic of a binary black hole system with circular orbits and (left) unequal, (right) equal, masses.

We can now write the mass density of the binary system as

$$T^{00}(t,\vec{x}) = \rho(r,\vec{x}) = M_1\delta^3(\vec{r}-\vec{r}_1) + M_2\delta^3(\vec{r}-\vec{r}_2), \tag{5.6}$$

where $\delta^3(\vec{r}-\vec{r}_1)$ is the Dirac delta function centered at position $\vec{r}_1$. Explicitly, the positions of the two black holes are

$$\vec{r}_1 = (r_1\cos\theta,\, r_1\sin\theta,\, 0), \quad \vec{r}_2 = -(r_2\cos\theta,\, r_2\sin\theta,\, 0), \tag{5.7}$$

where θ is the azimuthal angle for the first body, and is sometimes called the anomaly. Using Kepler's third law we can write the period P as

$$P^2 = \frac{4\pi^2}{M_1+M_2}(r_1+r_2)^3 \Rightarrow P = 2\pi\left(\frac{r^3}{M}\right)^{1/2}, \tag{5.8}$$

where $r = r_1 + r_2$ and $M = M_1 + M_2$. The angular velocity of the orbit is

$$\Omega = \frac{2\pi}{P} = \left(\frac{M}{r^3}\right)^{1/2}, \tag{5.9}$$

allowing us to write the positions of the two black holes as

$$\vec{r}_1 = (r_1\cos\Omega t,\, r_1\sin\Omega t,\, 0), \quad \vec{r}_2 = -(r_2\cos\Omega t,\, r_2\sin\Omega t,\, 0). \tag{5.10}$$

The density is now

$$\rho(t,\vec{x}) = M_1\delta(x-r_1\cos\Omega t)\delta(y-r_1\sin\Omega t)\delta(z) + M_2\delta(x-r_2\cos\Omega t)\delta(y-r_2\sin\Omega t)\delta(z). \tag{5.11}$$

The delta functions make the quadrupole moment Eq. (3.17) easy to calculate. The components are

$$\begin{aligned}
Q_{11} &= M_1\left(r_1^2\cos^2\Omega t - \frac{1}{3}r_1^2\right) + M_2\left(r_2^2\cos^2\Omega t - \frac{1}{3}r_2^2\right) \\
&= \left(\frac{M_1M_2^2}{(M_1+M_2)^2} + \frac{M_2M_1^2}{(M_1+M_2)^2}\right) r^2\left(\cos^2\Omega t - \frac{1}{3}\right) && \text{(by Eq. (5.3))} \\
&= \eta M r^2\left(\cos^2\Omega t - \frac{1}{3}\right) \\
&= \frac{1}{2}\eta M r^2\left(\frac{1}{3} + \cos(2\Omega t)\right). && (5.12)
\end{aligned}$$

Likewise

$$Q_{22} = \eta M r^2\left(\sin^2\Omega t - \frac{1}{3}\right) = \frac{1}{2}\eta M r^2\left(\frac{1}{3} - \cos(2\Omega t)\right), \tag{5.13}$$

$$Q_{33} = -\frac{1}{3}(M_1r_1^2 + M_2r_2^2) = -\eta M r^2/3. \tag{5.14}$$

Any off-diagonal terms involving z will vanish, thus the only surviving off-diagonal term is

$$\begin{aligned}
Q_{12} = Q_{21} &= M_1r_1^2\cos\Omega t\sin\Omega t + M_2r_2^2\cos\Omega t\sin\Omega t \\
&= \eta M r^2\cos\Omega t\sin\Omega t = \frac{1}{2}\eta M r^2\sin(2\Omega t). && (5.15)
\end{aligned}$$

Putting these together, we may construct the quadrupole moment tensor for a binary black hole system of unequal masses but circular orbits:

$$\boxed{Q_{ij} = \frac{1}{2}\eta M r^2\begin{pmatrix} \frac{1}{3}+\cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & \frac{1}{3}-\cos(2\Omega t) & 0 \\ 0 & 0 & -2/3 \end{pmatrix}.} \tag{5.16}$$

Taking two time derivatives of Eq. (5.16), we have

$$\ddot{Q}_{ij} = 2\eta M r^2 \Omega^2 \begin{pmatrix} -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ -\sin(2\Omega t) & \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.17}$$

With the quadrupole moment in hand, we can now write down the metric perturbation from the quadrupole formula Eq. (3.16), since the second time derivative of Q_{ij} and I_{ij} are identical:

$$\boxed{\tilde{h}_{ij} = \frac{4\eta M r^2 \Omega^2}{R} \begin{pmatrix} -\cos[2\Omega(t-r)] & -\sin[2\Omega(t-r)] & 0 \\ -\sin[2\Omega(t-r)] & \cos[2\Omega(t-r)] & 0 \\ 0 & 0 & 0 \end{pmatrix},} \tag{5.18}$$

where R is the observer's distance from the source (for the approximation to be valid, $R \gg r$). From this, we can see that the gravitational waves have a frequency twice that of the orbit of the black holes. Thus in each orbit, the black holes emit two full wavelengths of gravitational radiation. This is another reminder of the quadrupole nature of gravitational waves; when we average the luminosity over the period of the black holes, we will obtain two periods worth of gravitational radiation.

Taking a third time derivative of Eq. (5.16) gives

$$\dddot{Q}_{ij} = 4\eta M r^2 \Omega^3 \begin{pmatrix} \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.19}$$

Then

$$\dddot{Q}_{ij}\dddot{Q}^{ij} = \left(4\eta M r^2 \Omega^3\right)^2 \left(2\sin^2(2\Omega t) + 2\cos^2(2\Omega t)\right) = 32\eta^2 M^2 r^4 \Omega^6, \tag{5.20}$$

and from Eq. (4.18), the luminosity of gravitational waves from a binary black hole system may be determined:

$$\boxed{L_{GW} = \frac{32\eta^2 M^2 r^4 \Omega^6}{5} = \frac{32(M_1 M_2)^2 (M_1 + M_2)}{5r^5}.} \tag{5.21}$$

Eq. (5.21) gives the amount of energy leaving the binary system per unit time as the black holes orbit each other and produce gravitational radiation. Were one to have held on to the factor of G/c^4 on the right-hand side of the Einstein field equations, at this point in the game they would find the expression for the luminosity to be

$$L_{\rm GW} = \frac{32G^4}{5c^5} \frac{(M_1 M_2)^2 (M_1 + M_2)}{r^5}. \tag{5.22}$$

We reinstate G, $c \neq 1$ units for practical purposes. There are several key results that we can derive from this. First, this loss of energy results in the two black holes moving closer to one another, a process called inspiral. If we think of this energy as coming from the total energy of the black hole orbits, then by using the chain rule the rate of change of the separation may be found by

$$\frac{dr}{dt} = \frac{dr}{dE}\frac{dE}{dt} = -\left(\frac{2r^2}{GM_1M_2}\right)\frac{32G^4}{5c^5}\frac{(M_1M_2)^2(M_1+M_2)}{r^5} = -\frac{64G^3}{5c^5}\frac{M_1M_2(M_1+M_2)}{r^3}, \tag{5.23}$$

where we have used the virial theorem to obtain $E = -\frac{1}{2}U$ for a circular binary, where $U = GM_1M_2/r$ is the Newtonian gravitational potential energy. The negative sign in Eq. (5.23) indicates that the orbit is shrinking. It will continue to shrink until the black holes merge. The time it takes for this to happen given some initial orbital separation r_0 can be found by simply separating Eq. (5.23) and integrating:

$$\Delta t_{\rm merge} = \frac{1}{\alpha}\int_{r_0}^{0} r^3\, dr = -\frac{r_0^4}{4\alpha}, \quad \alpha \equiv -\frac{64G^3 M_1 M_2 (M_1 + M_2)}{5c^5}. \tag{5.24}$$

As the black holes spiral inwards, the angular frequency of the orbit increases. We define $\omega \equiv 1/P$ to be the orbital frequency of the binary, so then $f \equiv 2\omega$ is the frequency of the gravitational waves. From Kepler's third law we may relate f to r by

$$P^2 = \frac{1}{\omega^2} = \frac{4\pi^2}{GM} r^3 \Rightarrow f = 2\omega = \frac{1}{\pi}\sqrt{GM} r^{-3/2}. \tag{5.25}$$

Then, to find the rate at which the frequency increases, we may use the chain rule and Eqs. (5.23) and (5.25), yielding

$$\begin{aligned} \dot{f} &= \frac{df}{dr}\frac{dr}{dt} = -\frac{3}{2\pi}\frac{\sqrt{GM}}{r^{5/2}}\left[-\frac{64G^3}{5c^5}\frac{M_1 M_2 (M_1+M_2)}{r^3}\right] \\ &= \frac{96G^{7/2}}{5\pi c^5}\frac{M_1 M_2 (M_1+M_2)^{3/2}}{r^{11/2}} = \frac{96G^{7/2}}{5\pi c^5}\left(\frac{\pi^2}{GM}\right)^{11/6} f^{11/3} M_1 M_2 (M_1+M_2)^{3/2} \\ &= \frac{96\pi^{8/3}G^{5/3}}{5c^5}\frac{M_1 M_2}{(M_1+M_2)^{1/3}} f^{11/3}. \end{aligned} \tag{5.26}$$

We define the chirp mass [7] as

$$\mathcal{M} \equiv \eta^{3/5} M = \left(\frac{M_1^3 M_2^3}{M_1 + M_2}\right)^{1/5}, \tag{5.27}$$

so that we can write the change in the frequency as

$$\boxed{\dot{f} = \frac{96\pi^{8/3}}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} f^{11/3}.} \tag{5.28}$$

From Eq. (5.18) we can now write the amplitude of the gravitational wave as

$$h_0 = \frac{4G\eta M r^2 \Omega^2}{c^4 R} = \frac{4\eta}{c^4 R}\frac{(GM)^2}{r} = \frac{4(GM)^2\eta}{c^4 R}\left(\frac{\pi^2}{GM}\right)^{1/3} f^{2/3} \tag{5.29}$$

$$= \frac{4\pi^{2/3}G^{5/3}}{c^4 R}\left(\frac{M_1 M_2}{(M_1+M_2)^{1/3}}\right) f^{2/3}, \tag{5.30}$$

or, in terms of the chirp mass,

$$\boxed{h_0 = \frac{4\pi^{2/3}(G\mathcal{M})^{5/3}}{c^4 R} f^{2/3}.} \tag{5.31}$$

With the essential formulae describing the radiation of gravitational waves from a binary black hole system now at hand, we will apply our formula to the two cases in which gravitational waves have been detected on Earth so far.

5.1 The Hulse-Taylor Pulsar Binary

The first observational evidence of gravitational radiation came in 1974 by Joseph Taylor and Russel Hulse's observations of the binary neutron star system PSR 1913+16. Hulse and Taylor's observations involved careful measurements of the orbital period of the two stars, using the rapid pulsations of the neutron stars as a clock. The timings of the pulse when measured at Earth were sometimes sooner than expected, and sometimes later than expected. This could be explained if the pulsar were in orbit about a companion star (which also happens to be a neutron star), with the periodicity in the pulsar timings resultant from the orbital modulation. Hulse and Taylor collected painstakingly accurate measurements of the orbital period for several years, noticing that over time the orbit of the neutron stars was slowly contracting. As it turns out, the contraction is precisely in agreement with what is expected were the neutron stars emitting gravitational radiation as they orbited one another (the agreement is with the equivalent of Eq. (5.23), generalized to elliptical orbits). With decades of data

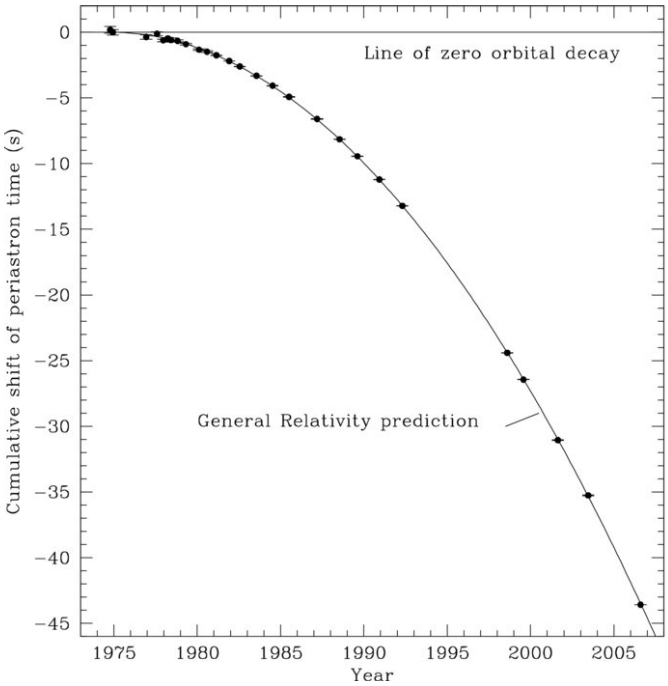

Figure 5.2: Over 30 years of measurements of the pulsar binary system PSR 1913+16. The shift in the periastron (position of closest approach to the companion star) is plotted on the vertical axis. This shirt corresponds to the shrinking of the orbit as the neutron stars spiral inwards, losing their orbital energy in the emission of gravitational waves. Just how much gravitational waves can be predicted from relativity (as we did in the simplified example of circular binaries) - the prediction precisely describes the observed data to within 0.2% [8].

now in our possession, the agreement of this system to general relativity has been measured to within 2%! [8]. The incredible agreement between the data and the prediction from relativity is displayed in Figure 5.2. This work earned Hulse and Taylor the 1993 Nobel Prize in Physics, and was the first indirect evidence supporting the existence of gravitational waves.

5.2 LIGO Observations of a Black Hole Merger

The Laser Interferometer Gravitational-Wave Observatory, or LIGO, is a pair of 4 km arm-length Michelson interferometers designed for the sole purpose of detecting gravitational waves. One of the twin interferometers is located in Hanford, Washington (LIGO Observatory 4k or LHO 4k, or simply H1), and the other in Livingston, Lousiana (LIGO Livingston Observatory 4k, or LLO 4k, or simply L1). The observatories rely in the principle of interferometry to detect gravitational waves: a high-power laser is sent from a source through a beam-splitter that sends half the beam down each of the 4 km arms of the interferometers, which are then each reflected by a mirror at the ends of the arms, and return back to the beam-splitter where the laser is recombined and directed to a photodetector that measures any interference pattern between the two beams. Should a gravitational wave pass through the experiment, one arm of the detector would be elongated and the other compressed, altering the light-travel-time of the two lasers in the arms, and thus altering the interference pattern at the photodetector. The ‘strain’ on the arms can be measured from this, although it is quite difficult: the arm-length only changes by a size much smaller than the diameter of an atomic nucleus!

Using equations we have already derived, we can estimate the shape of such a strain on the detector for the case of the binary black hole system previously discussed. Separating and integrating (5.28), we find the expression for the gravitational wave frequency to be

$$f(t) = \left(f_0^{-8/3} - \frac{256\pi^{8/3}}{5} \left(\frac{G\mathcal{M}}{c^3} \right)^{5/3} t \right)^{-3/8}, \tag{5.32}$$

where $f_0 = \Omega_0/\pi = \sqrt{GM/r_0^3}/\pi$ is the initial gravitational wave frequency. Then, with the expression (5.31) for the amplitude, we can create the waveform of the gravitational waves:

$$h(t) = h_0(t) \cos\left[2\pi f(t)\, t + \pi \dot{f}(t)\, t^2\right] \tag{5.33}$$

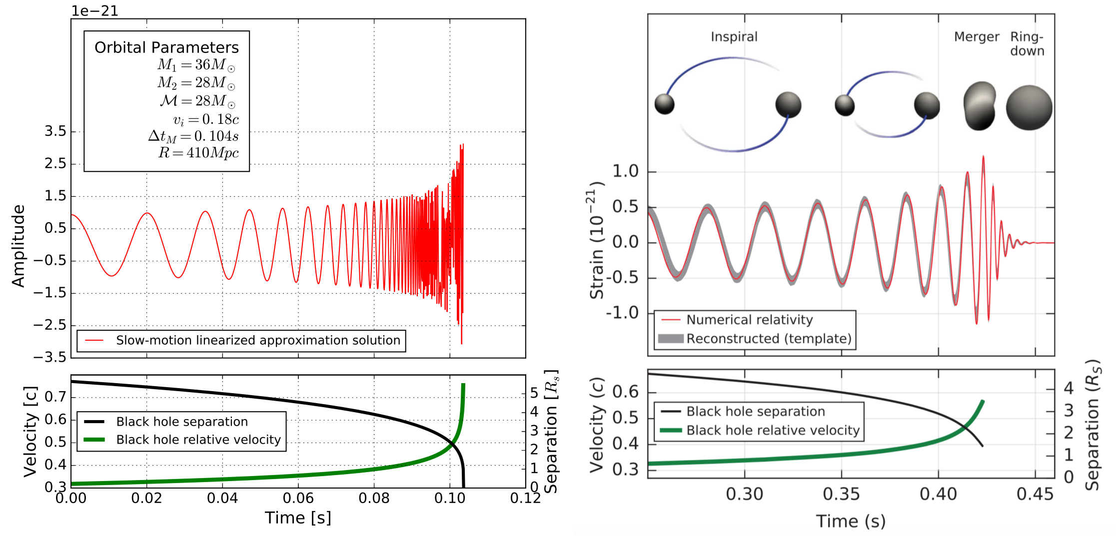

Figure 5.3: (Left) Plot of the binary black hole waveform as a function of time, as well as the black hole separation and relative velocity, all as calculated from our slow-motion approximation to the linearized theory of general relativity. (Right) Computer simulations solving Einstein's equations numerically for the waveform of coalescing binary black holes [9]. Note the strength of the velocities, especially right before the merger takes place (as well as the shape of the waveform) - here it becomes clear that our approximation is beginning to break down.

The waveform (5.33) describes the time evolution of the gravitational waves passing through the detector in terms of their amplitude (5.31), frequency (5.32), and rate of change of the frequency (5.28). The amplitude of the gravitational waves is proportional to the strain experienced by the interferometer arms elongating and contracting. To see what this waveform looks like, we use the measured parameters for the black holes observed in the event GW150914 at LIGO in September 2015 [9]. This was the first direct detection of gravitational waves. The parameters for the binary are:

$$M_1 = 36^{+5}_{-4} M_\odot, \quad M_2 = 29^{+4}_{-4} M_\odot$$
$$r_0 = 4R_s = \frac{8GM}{c^2}, \quad R = 410^{+160}_{-180}\ \text{Mpc}. \tag{5.34}$$

From these parameters, we find the chirp mass to be $\mathcal{M} = 28M_\odot$, the luminosity of the gravitational waves to be $L_{GW} = 4.3 \times 10^{54}$ erg/s ([9] finds $3.6^{+0.5}_{-0.4} \times 10^{56}$ erg/s using numerical relativity calculations), and the time to merge to be $\Delta t_{\text{merge}} = 0.104$ s. We can also use Eq. (5.23) to obtain the time evolution of the black hole separation and relative velocities. These quantities, along with the waveform, are shown in Figure 5.3 alongside the results obtained by [9] using more accurate numerical calculations. The observed data from both L1 and L2 is displayed in Figure 5.4. Our expressions for the waveforms can be seen to be remarkably accurate even at relatively high velocities (up to a third the speed of light just before merging!), despite our equations being derived from the assumption that the source is slow-moving.

The observations from LIGO, coupled with the discovery by Hulse and Taylor, present conclusive evidence that gravitational waves are in fact a physical property of our universe. Even today we are still finding new evidence matching the predictions of Einstein a hundred years ago. The general theory of relativity is truly an incredible achievement of the human mind to not only withstand this test of time, but also to match the experiments so beautifully. With the ability to make direct observations of gravitational waves now at our disposal, the field of gravitational wave astronomy has been inaugurated. With each discovery comes even more potential for the future. Perhaps we can use gravitational waves to see beyond the cosmic microwave background radiation and into the earliest moments after the big bang. Perhaps we may be able to probe the inner workings of black holes through the ripples in space they create. It is these prospects and more that make it an exciting time to be a part of humanity's quest to understand the universe.

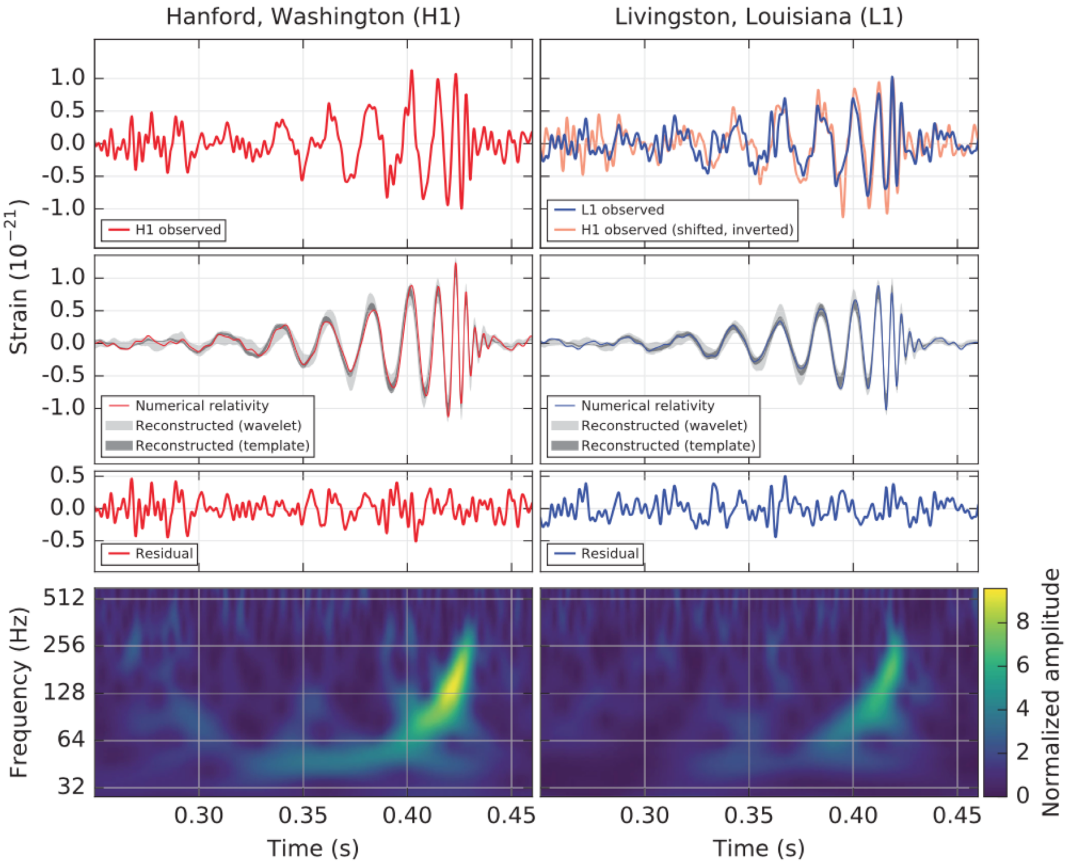

Figure 5.4: Observed strain on the interferometer arms due to the gravitational waves [9].

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A

The following is an exercise one should do once, and once only, in their life. Begin with the expression for the Christoffel symbols found in (2.4):

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\left(\partial_{\mu} h^{\sigma}_{\nu} + \partial_{\nu} h^{\sigma}_{\mu} - \eta^{\sigma\rho}\partial_{\rho} h_{\mu\nu}\right) - \frac{1}{2} h^{\sigma\rho}\left(\partial_{\mu} h_{\rho\nu} + \partial_{\nu} h_{\mu\rho} - \partial_{\rho} h_{\mu\nu}\right). \tag{A.1}$$

We substitute the Christoffel symbols directly into the definition of the Ricci curvature tensor,

$$R_{\rho\nu} = \partial_{\sigma}\Gamma^{\sigma}_{\rho\nu} - \partial_{\nu}\Gamma^{\sigma}_{\rho\sigma} + \Gamma^{\sigma}_{\alpha\sigma}\Gamma^{\alpha}_{\rho\nu} - \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\rho\sigma}, \tag{A.2}$$

yielding

$$\begin{aligned} R_{\rho\nu} = &\frac{1}{2}\left[\partial_{\sigma}\partial_{\rho} h^{\sigma}_{\nu} + \partial_{\sigma}\partial_{\nu} h^{\sigma}_{\rho} - \partial_{\sigma}\eta^{\sigma\lambda}\partial_{\lambda} h_{\rho\nu}\right] - \frac{1}{2}\partial_{\sigma}\left[h^{\sigma\lambda}\left(\partial_{\rho} h_{\lambda\nu} + \partial_{\nu} h_{\rho\lambda} - \partial_{\lambda} h_{\rho\nu}\right)\right] \\ &- \frac{1}{2}\left[\partial_{\nu}\partial_{\rho} h^{\sigma}_{\sigma} + \partial_{\nu}\partial_{\sigma} h^{\sigma}_{\rho} - \partial_{\nu}\eta^{\sigma\lambda}\partial_{\lambda} h_{\rho\sigma}\right] + \frac{1}{2}\partial_{\nu}\left[h^{\sigma\lambda}\left(\partial_{\rho} h_{\lambda\sigma} + \partial_{\sigma} h_{\rho\lambda} - \partial_{\lambda} h_{\rho\sigma}\right)\right] \\ &+ \frac{1}{4}\left(\partial_{\alpha} h^{\sigma}_{\sigma} + \partial_{\sigma} h^{\sigma}_{\alpha} - \eta^{\sigma\lambda}\partial_{\lambda} h_{\alpha\sigma}\right)\left(\partial_{\rho} h^{\alpha}_{\nu} + \partial_{\nu} h^{\alpha}_{\rho} - \eta^{\alpha\lambda}\partial_{\lambda} h_{\rho\nu}\right) + \mathcal{O}(h^3) + \mathcal{O}(h^4) \\ &- \frac{1}{4}\left(\partial_{\alpha} h^{\sigma}_{\nu} + \partial_{\nu} h^{\sigma}_{\alpha} - \eta^{\sigma\lambda}\partial_{\lambda} h_{\alpha\nu}\right)\left(\partial_{\rho} h^{\alpha}_{\sigma} + \partial_{\sigma} h^{\alpha}_{\rho} - \eta^{\alpha\lambda}\partial_{\lambda} h_{\rho\sigma}\right) + \mathcal{O}(h^3) + \mathcal{O}(h^4). \end{aligned} \tag{A.3}$$

We drop the terms of third order and higher, and group terms by first and second order. All the first order terms are contained in the first and third terms with the square brackets. We write these as

$$\begin{aligned} R^{(1)}_{\rho\nu} &= \frac{1}{2}\left(\partial_{\sigma}\partial_{\rho} h^{\sigma}_{\nu} + \cancel{\partial_{\sigma}\partial_{\nu} h^{\sigma}_{\rho}} - \partial_{\sigma}\partial^{\sigma} h_{\rho\nu} - \partial_{\nu}\partial_{\rho} h - \cancel{\partial_{\nu}\partial_{\sigma} h^{\sigma}_{\rho}} + \partial_{\nu}\partial_{\sigma} h^{\sigma}_{\rho}\right) \\ &= \frac{1}{2}\left(\partial_{\sigma}\partial_{\rho} h^{\sigma}_{\nu} - \partial_{\sigma}\partial^{\sigma} h_{\rho\nu} - \partial_{\nu}\partial_{\rho} h + \partial_{\nu}\partial_{\sigma} h^{\sigma}_{\rho}\right). \end{aligned} \tag{A.4}$$

This is the expression (2.7). Looking to second order, we have

$$\begin{aligned} R^{(2)}_{\rho\nu} = &\frac{1}{2}\Big(\underline{\partial_{\nu} h^{\sigma\lambda}\partial_{\rho} h_{\lambda\sigma}} + \cancel{\partial_{\nu} h^{\sigma\lambda}\partial_{\sigma} h_{\rho\lambda}} - \cancel{\partial_{\nu} h^{\sigma\lambda}\partial_{\lambda} h_{\rho\sigma}} + h^{\sigma\lambda}\partial_{\nu}\partial_{\rho} h_{\lambda\sigma} + \cancel{h^{\sigma\lambda}\partial_{\nu}\partial_{\sigma} h_{\rho\lambda}} - h^{\sigma\lambda}\partial_{\nu}\partial_{\lambda} h_{\rho\sigma} \\ &- \partial_{\sigma} h^{\sigma\lambda}\partial_{\rho} h_{\lambda\nu} - \partial_{\sigma} h^{\sigma\lambda}\partial_{\nu} h_{\rho\lambda} + \partial_{\sigma} h^{\sigma\lambda}\partial_{\lambda} h_{\rho\nu} - h^{\sigma\lambda}\partial_{\sigma}\partial_{\rho} h_{\lambda\nu} - \cancel{h^{\sigma\lambda}\partial_{\sigma}\partial_{\nu} h_{\rho\lambda}} + h^{\sigma\lambda}\partial_{\sigma}\partial_{\lambda} h_{\rho\nu}\Big) \\ + &\frac{1}{4}\Big(\partial_{\alpha} h\partial_{\rho} h^{\alpha}_{\nu} + \partial_{\alpha} h\partial_{\nu} h^{\alpha}_{\rho} - \partial_{\alpha} h\partial^{\alpha} h_{\rho\nu} + \cancel{\partial_{\sigma} h^{\sigma}_{\alpha}\partial_{\rho} h^{\alpha}_{\nu}} + \cancel{\partial_{\sigma} h^{\sigma}_{\alpha}\partial_{\nu} h^{\alpha}_{\rho}} - \cancel{\partial_{\sigma} h^{\sigma}_{\alpha}\partial^{\alpha} h_{\rho\nu}} \\ &- \cancel{\partial_{\sigma} h^{\sigma}_{\alpha}\partial_{\rho} h^{\alpha}_{\nu}} - \cancel{\partial_{\sigma} h^{\sigma}_{\alpha}\partial_{\nu} h^{\alpha}_{\rho}} + \cancel{\partial_{\sigma} h^{\sigma}_{\alpha}\partial^{\alpha} h_{\rho\nu}} - \cancel{\partial_{\alpha} h^{\sigma}_{\nu}\partial_{\rho} h^{\alpha}_{\sigma}} - \underline{\partial_{\alpha} h^{\sigma}_{\nu}\partial_{\sigma} h^{\alpha}_{\rho}} + \underline{\partial_{\alpha} h^{\sigma}_{\nu}\partial^{\alpha} h_{\rho\sigma}} \\ &- \underline{\partial_{\nu} h^{\sigma}_{\alpha}\partial_{\rho} h^{\alpha}_{\sigma}} - \cancel{\partial_{\nu} h^{\sigma}_{\alpha}\partial_{\sigma} h^{\alpha}_{\rho}} + \cancel{\partial_{\nu} h^{\sigma}_{\alpha}\partial^{\alpha} h_{\rho\sigma}} + \cancel{\partial^{\sigma} h_{\alpha\nu}\partial_{\rho} h^{\alpha}_{\sigma}} + \underline{\partial^{\sigma} h_{\alpha\nu}\partial_{\sigma} h^{\alpha}_{\rho}} - \underline{\partial^{\sigma} h_{\alpha\nu}\partial^{\alpha} h_{\rho\sigma}}\Big), \end{aligned} \tag{A.5}$$

where underlining indicates that the terms of corresponding color combine, and the slashes indicate cancellations. Terms without any colored indicator are unaffected. The surviving terms are

$$\begin{aligned} R^{(2)}_{\rho\nu} = &\frac{1}{2}\Big(h^{\sigma\lambda}\partial_{\rho}\partial_{\nu} h_{\sigma\lambda} + h^{\sigma\lambda}\partial_{\sigma}\partial_{\lambda} h_{\rho\nu} - h^{\sigma\lambda}\partial_{\lambda}\partial_{\nu} h_{\rho\sigma} - h^{\sigma\lambda}\partial_{\lambda}\partial_{\rho} h_{\nu\sigma} \\ &+ \partial_{\sigma} h^{\sigma\lambda}\partial_{\lambda} h_{\rho\nu} + \partial^{\sigma} h^{\lambda}_{\nu}\partial_{\sigma} h_{\lambda\rho} - \partial^{\sigma} h^{\lambda}_{\nu}\partial_{\lambda} h_{\sigma\rho} - \partial_{\sigma} h^{\sigma\lambda}\partial_{\nu} h_{\rho\lambda} - \partial_{\sigma} h^{\sigma\lambda}\partial_{\rho} h_{\lambda\nu}\Big) \\ + &\frac{1}{4}\Big(\partial_{\rho} h_{\sigma\lambda}\partial_{\nu} h^{\sigma\lambda} + \partial_{\lambda} h\partial_{\rho} h^{\lambda}_{\nu} + \partial_{\lambda} h\partial_{\nu} h^{\lambda}_{\rho} - \partial_{\lambda} h\partial^{\lambda} h_{\rho\nu}\Big), \end{aligned} \tag{A.6}$$

which is the expression (4.3).

B

In this appendix we calculate Eq. (4.7) given Eq. (4.3) in order to obtain an expression for $t_{\rho\nu}$. First, we write $\left\langle R^{(2)}_{\rho\nu} \right\rangle$ from the definition Eq. (4.3):

$$\begin{aligned}\left\langle R^{(2)}_{\rho\nu} \right\rangle = \frac{1}{2}\Big[&\left\langle h^{\sigma\lambda}\partial_\rho\partial_\nu h_{\sigma\lambda} \right\rangle + \left\langle h^{\sigma\lambda}\partial_\sigma\partial_\lambda h_{\rho\nu} \right\rangle - \left\langle h^{\sigma\lambda}\partial_\lambda\partial_\nu h_{\rho\sigma} \right\rangle - \left\langle h^{\sigma\lambda}\partial_\lambda\partial_\rho h_{\nu\sigma} \right\rangle \\ &+ \left\langle \partial_\sigma h^{\sigma\lambda}\partial_\lambda h_{\rho\nu} \right\rangle + \left\langle \partial^\sigma h^\lambda_\nu \partial_\sigma h_{\lambda\rho} \right\rangle - \left\langle \partial^\sigma h^\lambda_\nu \partial_\lambda h_{\sigma\rho} \right\rangle - \left\langle \partial_\sigma h^{\sigma\lambda}\partial_\nu h_{\rho\lambda} \right\rangle - \left\langle \partial_\sigma h^{\sigma\lambda}\partial_\rho h_{\lambda\nu} \right\rangle \Big] \\ + \frac{1}{4}\Big[&\left\langle \partial_\rho h_{\sigma\lambda}\partial_\nu h^{\sigma\lambda} \right\rangle + \left\langle \partial_\lambda h \partial_\rho h^\lambda_\nu \right\rangle + \left\langle \partial_\lambda h \partial_\nu h^\lambda_\rho \right\rangle - \left\langle \partial_\lambda h \partial^\lambda h_{\rho\nu} \right\rangle \Big] .\end{aligned} \tag{B.1}$$

Next we simplify this expression making use the Lorenz and TT gauge conditions, as well as the fact that we may integrate by parts in the averages to write

$$\langle \partial_\mu X \partial_\nu Y \rangle = -\langle X \partial_\mu \partial_\nu Y \rangle , \tag{B.2}$$

since the boundary term vanishes. Now, the only reason that it is okay to use these gauge conditions to simplify the calculation is that, in hindsight, the final result actually ends up being gauge invariant [6]. The TT gauge conditions immediately kill off any of the terms involving traces (the last three terms), and the Lorenz condition kills any terms including a divergence (terms 2, 3, 4, 5, 8, and 10). We also notice that we may use integration by parts to flip the derivatives of the very first term in the (1/2) square bracket, allowing us to combine it with the first term in the (1/4) square bracker. Just by using these conditions, we reduce Eq. (B.1) to

$$\left\langle R^{(2)}_{\rho\nu} \right\rangle = -\frac{1}{4}\left\langle \partial_\rho h_{\sigma\lambda}\partial_\nu h^{\sigma\lambda} \right\rangle - \frac{1}{2}\left\langle h^\lambda_\nu \partial_\sigma \partial^\sigma h_{\rho\lambda} \right\rangle - \frac{1}{2}\left\langle \partial^\lambda h_{\lambda\nu} \partial^\sigma h_{\sigma\rho} \right\rangle , \tag{B.3}$$

where the second term was obtained by using integration by parts on the sixth term of Eq. (B.1), and the third term above is obtained by using integration by parts twice to flip both derivatives in the seventh term of Eq. (B.1). Writing it this way, it is easy to see that the Lorenz gauge kills this third term. Recall that in the TT gauge $h_{\mu\nu} = \tilde{h}_{\mu\nu}$, thus the second term vanishes because $h_{\mu\nu}$ satisfies the wave equation very far from the source where $\Box h_{\mu\nu} = 0$. Thus, all that remains is the first term:

$$\left\langle R^{(2)}_{\rho\nu} \right\rangle = -\frac{1}{4}\left\langle \partial_\rho h_{\sigma\lambda}\partial_\nu h^{\sigma\lambda} \right\rangle . \tag{B.4}$$

If we take the trace of Eq. (B.4), we could integrate by parts to obtain another wave equation that will vanish. Thus Eq. (4.7) becomes, quite simply

$$t_{\rho\nu} = -\frac{1}{8\pi}\left\langle R^{(2)}_{\rho\nu} \right\rangle , \tag{B.5}$$

and thus the result, expressing all quantities as explicitly being evaluated in the TT gauge, is

$$t_{\rho\nu} = \frac{1}{32\pi}\left\langle \partial_\rho h^{\mathrm{TT}}_{\sigma\lambda}\partial_\nu h^{\sigma\lambda}_{\mathrm{TT}} \right\rangle , \tag{B.6}$$

which is the expression (4.8)